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Buy-price English auction

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Abstract

Consider an English auction for a single object in which there is an option for a bidder to guarantee a purchase at a seller-specified buy price b at any time. We show that there exist $\tilde{v}$ and $\hat{v}$ ($\geqslant \tilde{v}$), such that a bidder purchases at the buy price immediately if his valuation v is no less than $\hat{v}$ or $\tilde{v} \leqslant v < \hat{v}$ and at least one other bidder is participating in the auction. If $b \leqslant v < \tilde{v}$, he purchases at the buy price once the current bid reaches a strategically chosen threshold price. A properly set buy price increases expected social welfare and the expected utility of each agent when either buyers or seller are risk-averse.

JEL classification: D44

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1. Introduction

The popularity, scope, and competitiveness of online auctions have encouraged auctioneers to innovate. Particularly noteworthy is the use of a buy price in English auctions where the seller announces a maximum bid level, at which any bidder can immediately win the

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auction. Since the start of such auctions in 1999, [1] there has emerged a significant portion of sellers who choose to utilize buy prices [6,9]. [2] For example, our preliminary study of over seven thousand sports rookie card auctions at Yahoo! suggests that about half of the auctions utilized buy prices and that approximately one-fourth of them ended with bidders exercising that option.

Wang [13] shows that an auction yields higher expected seller revenue than a posted-price sale when the auction is costless. Auctioning online is very affordable, and its popularity indicates that many sellers have recognized the superiority of auctions over posted-price sales. Then why would the seller prefer to specify a posted price when dynamic pricing is in play and consequently restrict her maximum payment? Should bidders also favor buy prices? If so, what are their equilibrium strategies for such auctions?

Before addressing these issues, we need to clarify the categories of buy prices. Currently, there are three types of buy prices: "permanent," "temporary," and "limited." A permanent buy price remains valid during the entire course of the auction. Yahoo!'s "buy now," uBid's "uBuy It," and Amazon's "Take-It" prices all fall into this category. Notably missing from this list is eBay, which offers only a temporary "buy-it-now" feature that disappears as soon as any bid is made at or above the reserve price. A buy price is limited if it is valid only for a restricted period of time during the auction. [3]

This paper focuses on the study of permanent buy prices because they allow for an ultimately hybrid model combining posted-price sales and auctions. For simplicity, we generally refer to the "permanent" buy price as the buy price in the following discussions, and we specify other types explicitly.

Budish and Takeyama [3] are the first to have recognized the benefits of buy prices. Using a simple two-bidder, two-value model, they conclude that with a properly set buy price, the seller facing risk-neutral buyers earns the same expected profit as in a standard English auction, but the seller facing risk-averse buyers gains higher expected profit. Reynolds and Wooders [9] confirm this result in a model of two bidders with uniform distribution.

But Budish and Takeyama [3] express doubts about the extension of their results to a general setting. They conjecture that "in a more general framework with n valuations, the optimal buy price may be less than the second-highest valuation, which admits the possibility of inefficient outcomes. In this case, revenue equivalence breaks down and the effectiveness of the buy price to enhance sellers' profits when bidders are risk-averse may be diminished."

We show that Budish and Takeyama's doubts are unfounded. We prove that, in a setting of n bidders with arbitrary continuous value distribution, if a buy price is properly set, revenue equivalence still holds when agents are risk-neutral, and the buy price still enhances sellers'

[1] Yahoo! started offering auction sellers the option to utilize buy prices in 1999 whereas eBay implemented its version of the buy price practice in November 2000.

[2] The list of data on the proportion of auctions with buy prices: eBay 2001 data shows 30% in 1Q 2001, 35% in 2Q 2001, and 45% in December 2001 [6]. 40% on eBay and 66% on Yahoo! in 2002 [9]. 37% on Bid or Buy in 2001 [6].

[3] For example, labx.com, specializing in the auctions of scientific equipment, requires that "an Auction Stop bid must be entered 48 hours prior to the auction close or the Auction Stop feature is dropped and the auction continues to the ending date specified."

profits when bidders are risk-averse. We further prove that if neither sellers nor bidders are risk takers, an English auction augmented with a properly set buy price weakly dominates the standard English auction. And more surprisingly, a buy-price English auction not only increases the expected social welfare, but also ensures that the expected utility of each agent is never lower than it is in a standard auction. Particularly when either the seller or buyers are risk-averse, the seller's expected utility is strictly higher than that in a standard auction and without lowering the buyers' expected utilities.

Utilizing buy prices in auctions can be viewed as providing a form of insurance for risk-averse buyers. If these buyers' valuations are above the buy price, they can bid the buy price to achieve a fixed profit instead of taking the risk of losing the item when bidding below the buy price. These buyers would pay premiums for avoiding the risk. Similar to how insurance companies make money, the seller utilizing buy prices profits by exploiting the risk aversion of these buyers. But unlike insurance, the seller does not have to be risk-neutral to benefit from buy prices; a risk-averse seller can gain even more because utilizing buy prices reduces the variance of seller revenue.

To prove the superiority of a buy-price English auction, we need to define buyers' equilibrium strategies. We will prove that a unique Bayesian Nash equilibrium exists for buyers when there are unique reference points $\tilde{v}$ and $\hat{v}$ ($\tilde{v} \leqslant \hat{v}$, and both above the buy price) so that a bidder with a valuation between the buy price and $\tilde{v}$ is a threshold bidder who exercises the buy price once the current high bid reaches a strategically chosen threshold price, a bidder with a valuation between $\tilde{v}$ and $\hat{v}$ is a conditional bidder who bids the buy price immediately on the condition that at least one competing bidder bids at or above the reserve price, and a bidder with a valuation above $\hat{v}$ is an unconditional bidder who selects to purchase at the buy price instantly with no conditions. We will prove that a lower bound exists so that if the buy price is at or above this bound, then all bidders with valuations above the buy price are threshold bidders. In this case the auction is efficient; it guarantees that the bidder with the highest valuation wins, because a bidder with a higher valuation has a lower threshold price. Moreover, the more risk-averse a bidder, the lower his threshold price. The seller thus has higher expected utility from risk-averse buyers than from risk-neutral buyers.

The paper proceeds as follows. In Section 2, we lay out the model, state and prove the bidders' unique Bayesian Nash equilibrium strategies. We also compare behaviors of bidders with different degrees of risk aversion. In Section 3, we prove that both risk-averse and risk-neutral bidders are not worse off in a buy-price English auction. We analyze the impact of the seller's risk preference on the use of buy prices and prove that the seller is never worse off utilizing properly set buy prices. We also derive the lower bound of a properly set buy price. Section 4 provides more intuitions about our results and recommends future research directions.

2. Bidders' equilibrium strategy

2.1. *The model*

There is one seller and n bidders in a buy-price English auction of an indivisible good. Only bids at or above the reserve price are valid, and the seller has committed not to relist the item if no valid bid emerges. This no-resale constraint is a standard assumption and is naturally satisfied in cases of perishable or time-sensitive goods like flowers or tickets. We also assume bidders have independent private valuations. This assumption is restrictive but is closely emulated by auctions of collectibles or used goods. Our empirical study also supports such an assumption because the data show that most sports rookie cards purchased through auctions are for collection rather than for resale—buyers seldom resell the cards they have just purchased.

To simplify the analysis, we use a "modified English clock auction" as our model, which has a set of rules as follows:

- The seller announces both a reserve price and a buy price before the auction starts. The auction starts at a pre-announced time with the auction clock being set at the reserve price. Each bidder controls two buttons: a "bid" button and a "buy" button. A bidder signals his willingness to pay the current clock price by pressing and holding down his "bid" button. Once a bidder releases his "bid" button, he quits the auction and can no longer return. A bidder does not know how many other bidders participate in the auction. At any time, a bidder can press his "buy" button signaling that he bids the buy price. A bidder starts signaling his actions shortly before the auction starts.
- At the start of the auction, the auctioneer checks the state of bidders' buttons. If only one "buy" button is pressed, the auction ends and the bidder who has pressed his "buy" button wins, paying the buy price. If more than one "buy" button is pressed, the winner is randomly chosen among those who have pressed their "buy" buttons. If neither the "buy" nor the "bid" button is pressed, the auction ends without a sale. If there is no "buy" button being pressed but one "bid" button is being held, the auction ends and the bidder who holds his "bid" button wins, paying the reserve price. If there is no "buy" button being pressed but more than one "bid" button being held, the auction clock starts ascending from the reserve price.
- The auction terminates when one of the following scenarios occur: (1) There is only one bidder left holding his "bid" button. This bidder wins and pays the current clock price. (2) There is a bidder who has pressed his "buy" button. This bidder wins, paying the buy price. If multiple bidders press their "buy" buttons simultaneously, the winner is chosen randomly among them. (3) The auction clock reaches the buy price. The winner is chosen randomly among the bidders who hold their "bid" buttons, and the winner pays the buy price.

2.2. *Proof of bidders' equilibrium strategies*

In a buy-price English auction, for a bidder with a valuation below or equal to the buy price, his pure and dominant strategy remains the same as in a standard English auction, i.e., to bid up to his valuation. However, the strategy space for a bidder with a valuation above the buy price becomes more complicated. When there are multiple bidders with such valuations, the winner will be the one who first commits to the buy price. If such a bidder thinks that at least one other bidder exists who might bid the buy price, he would find the appropriate moment to bid the buy price before any other bidder. If he thinks that there is no other bidder who might use the buy price, he would simply keep bidding. Consequently, there is no dominant strategy for such a bidder; we could only hope to find a Bayesian Nash equilibrium.

To find such an equilibrium, we first need to characterize all possible pure strategies that a bidder can follow:

- *Traditional*: Bid up to his valuation;
- *Threshold*: Keep bidding until winning or his threshold price is reached. Once the auction clock reaches his threshold price, bid the buy price immediately.
- *Conditional*: Bid, but use the buy price immediately if at least one other bidder bids at or above the reserve price.
- *Unconditional*: Bid the buy price immediately with no conditions.

We can unite the above four strategies under a "generalized threshold strategy" in which each bidder has a threshold price that determines if and when the bidder uses the buy price. If the buy price is above a bidder's valuation, he never uses the buy price and hence we can assume he has a threshold price above his valuation. We can regard that a bidder following the traditional strategy as having a threshold price equal to the buy price. A bidder following the threshold strategy has a threshold price dependent on his valuation and the buy price (as we will show later). A bidder following the conditional strategy can be regarded as having a threshold price equal to the reserve price,[4] and a bidder following the unconditional strategy can be regarded as having a threshold price less than the reserve price, say, the lowest bidder valuation.

Let r denote the reserve price and b the buy price. Let us assume bidders' valuations are drawn randomly from the same cumulative probability distribution F, which is strictly increasing and differentiable over the support of bidder valuations, $[\underline{v}, \overline{v}]$. Let $f = F'$ denote the probability density. Let $u(x)$ denote the bidder's von Neumann–Morgenstern utility function, where x is the difference between the buyer's valuation and his payment if he wins and is zero if otherwise. Let $s(p)$ denote the seller's utility when she sells the item and receives the payment p. Both $u(x)$ and $s(p)$, we assume, are linear or concave, twice continuously differentiable, and strictly increasing. Let v_s be the seller's valuation for the item, assuming $u(0) = s(v_s) = 0$ and $u'(0) = s'(v_s) = 1$.

The following theorem defines the bidders' unique Bayesian Nash equilibrium (see Fig. 1):

Theorem 1. *A buy-price English auction has a unique Bayesian Nash equilibrium determined by constants $\tilde{v}$ and $\hat{v}$ and function t, $b < \tilde{v} \leqslant \hat{v} \leqslant \overline{v}$, such that all bidders with valuation v have the following strategies*:

- *Use the traditional strategy if $v < b$.*

[4] Conditional bidders use the buy price immediately upon learning that they have competition thus they cannot obtain the item at the reserve price. They are different from unconditional bidders, because unconditional bidders give up the chance to obtain the item at the reserve price, but in exchange, they are guaranteed to win over any conditional bidders. Conditional bidders observe the auction clock and bid the buy price as soon as they notice the auction clock departing from the reserve.

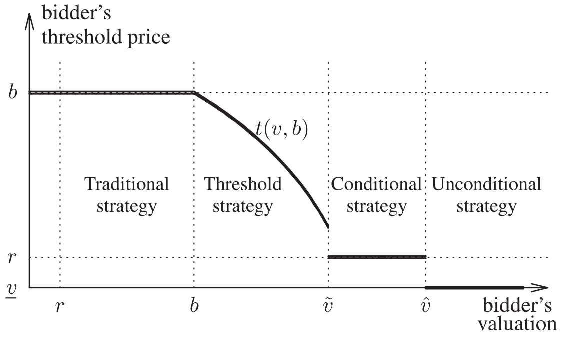

Fig. 1. Bidders in a buy-price English auction follow one of the four equilibrium strategies dependent on their valuations.

- *Use the threshold strategy with a threshold price* $t(v, b) \in (r, b]$ *if* $b \leqslant v < \tilde{v}$, *where* $t(v, b)$ *is the threshold function defined by the differential equation*

$$\frac{u(v - t(v,b))}{u(v-b)} - 1 + \frac{F^{n-1\prime}(v)}{\partial_1 (F^{n-1} \circ t)(v,b)} = 0,$$
$$\partial_1 t(b,b) = -1, \quad t(b,b) = b.$$

- *Use the conditional strategy if* $\tilde{v} \leqslant v < \hat{v}$.
- *Use the unconditional strategy if* $v \geqslant \hat{v}$.

All bidders with $v \geqslant b$ *follow the threshold strategy, i.e.,* $\tilde{v} = \hat{v} = \overline{v}$, *if and only if* $\lim_{v \to \overline{v}} t(v, b) \geqslant r$.

Proof. We need to obtain the necessary and sufficient conditions for t, $\tilde{v}$, and $\hat{v}$ to determine a symmetric Bayesian Nash equilibrium. First we assume t, $\tilde{v}$, and $\hat{v}$ determine a pure strategy equilibrium. Under this assumption we calculate bidders' expected profits corresponding to the different strategies, prove $t(v, b)$ is both strictly decreasing and continuous in v when $b \leqslant v < \tilde{v}$, and show how to compute $t(v, b)$. We then prove such a $t(v, b)$ indeed corresponds to a unique symmetric Bayesian Nash equilibrium. Further, we prove the existence of unique reference points $\tilde{v}$ and $\hat{v}$ and show how to compute them.

Since we are concerned with strategies for a bidder with $v \geqslant b$, let us first calculate such a bidder's expected profit under the three different strategies:

1. *Under the unconditional strategy*: A bidder using the unconditional strategy competes only with other bidders using the same strategy. The probability that a bidder uses the unconditional strategy is $1 - F(\hat{v})$, and the probability that there are exactly $k - 1$ other bidders ($1 \leqslant k \leqslant n$) who also use the unconditional strategy is $\binom{n-1}{k-1} (1 - F(\hat{v}))^{k-1} F^{n-k}(\hat{v})$.

Hence, an unconditional bidder's expected profit, denoted by $\Pi_{\mathrm{u}}(v)$, is

$$\Pi_{\mathrm{u}}(v) = \sum_{k=1}^{n} \binom{n-1}{k-1} (1 - F(\hat{v}))^{k-1} F^{n-k}(\hat{v}) \frac{u(v-b)}{k}$$
$$= \frac{1 - F^n(\hat{v})}{n(1 - F(\hat{v}))} u(v-b) = \frac{u(v-b)}{n} \sum_{k=0}^{n-1} F^k(\hat{v}). \quad (1)$$

2. *Under the conditional strategy*: A bidder using the conditional strategy wins if (1) there are no unconditional bidders and (2) he is chosen randomly among the conditional bidders who bid the buy price simultaneously. A conditional bidder pays b if there is another valid bid and pays r if there are no valid bids. Hence, a conditional bidder's expected profit, denoted by $\Pi_{\mathrm{c}}(v)$, is

$$\Pi_{\mathrm{c}}(v) = \sum_{k=2}^{n} \binom{n-1}{k-1} (F(\hat{v}) - F(\tilde{v}))^{k-1} F^{n-k}(\tilde{v}) \frac{u(v-b)}{k}$$
$$+ u(v-b)(F^{n-1}(\tilde{v}) - F^{n-1}(r)) + u(v-r)F^{n-1}(r)$$
$$= \frac{u(v-b)}{n} \sum_{k=0}^{n-1} F^k(\hat{v}) F^{n-k-1}(\tilde{v}) + (u(v-r) - u(v-b))F^{n-1}(r). \quad (2)$$

3. *Under the threshold strategy*: A bidder with a threshold price p wins and pays b if there are neither unconditional nor conditional bidders and he is chosen randomly among the threshold bidders who bid the buy price simultaneously once the current high bid reaches p. Alternatively, he wins and pays the second-highest bid (or the reserve if he is the only bidder with a valuation at or above the reserve) if all other bidders have valuations below p. Let $G_{n-1}(p)$ be the probability that a bidder with a threshold price p exercises the buy price and wins the auction.[5] A threshold bidder's expected profit, denoted by $\Pi_{\mathrm{t}}(v, p)$, is

$$\Pi_{\mathrm{t}}(v, p) = u(v-b)G_{n-1}(p) + \int_r^p u(v-x)\, \mathrm{d}F^{n-1}(x) + u(v-r)F^{n-1}(r). \quad (3)$$

We now establish two properties of the threshold function $t(v, b)$. To simplify the notations in the following proofs, we define t for the full range of bidder valuations: $t(v, b) = b$ when $v \leqslant b$, $t(v, b) = r$ when $\tilde{v} \leqslant v < \hat{v}$, and $t(v, b) = \underline{v} < r$ when $v \geqslant \hat{v}$.

Proposition 2. *$t(v, b)$ is strictly decreasing in v for $b \leqslant v < \tilde{v}$.*

Proof. See Appendix A. □

Proposition 2 implies that for any threshold price p, $r < p \leqslant b$, there is at most one bidder valuation corresponding to the equilibrium threshold price p; i.e., there is at most one v s.t., $t(v, b) = p$. Thus in equilibrium it is impossible for two threshold bidders to have the same

[5] If multiple bidders use the same threshold price p, the winner is selected randomly among them. This will be reflected in $G_{n-1}(p)$. Later we prove that in equilibrium there is zero probability that multiple threshold bidders share the same threshold price.

threshold price. This allows us to express $G_{n-1}(p)$, which denotes the probability that a bidder with threshold price p uses the buy price and wins. It occurs if no one else has a threshold price lower than p, excluding the case where the bidder wins without using the buy price because everyone else has a valuation less than p:

$$G_{n-1}(p) = (1 - \text{Prob}(t(v, b) \leqslant p))^{n-1} - F^{n-1}(p).$$

Defining $T(p) = \text{Prob}(t(v, b) \leqslant p)$, we can write $G_{n-1}(p)$ as

$$G_{n-1}(p) = (1 - T(p))^{n-1} - F^{n-1}(p). \tag{4}$$

Note that $t(v, b)$ strictly decreasing for $b \leqslant v < \tilde{v}$ implies that $T(p)$ is continuous and strictly increasing for $r < p \leqslant b$, which in turn implies that $G_{n-1}(p)$ is continuous and strictly decreasing for $r < p \leqslant b$. This can be used to prove the following proposition:

Proposition 3. *$t(v, b)$ is continuous in v for $b \leqslant v < \tilde{v}$.*

Proof. See Appendix B. □

Note that the equilibrium threshold function cannot continuously drop to the reserve price if there is a positive probability that there are conditional bidders. This is because a bidder with a threshold slightly above the reserve can switch to the conditional strategy, thus increasing his chance of winning by more than a fixed positive amount while his payment conditional on winning would hardly change.

Using the above two propositions we can now have the following proposition. [6]

Proposition 4. *Any threshold price function $t(v, b)$ corresponding to an equilibrium satisfies the following differential equation*:

$$\frac{u(v - t(v, b))}{u(v - b)} - 1 + \frac{F^{n-1\prime}(v)}{\partial_1 (F^{n-1} \circ t)(v, b)} = 0 \quad \text{if } b \leqslant v < \tilde{v}, \tag{5}$$
$$\partial_1 t(b, b) = -1, \quad t(b, b) = b.$$

Proof. In equilibrium, when all other bidders follow the threshold strategy determined by t, the optimal threshold price of a bidder with valuation v ($b \leqslant v < \tilde{v}$) is $t(v, b)$, i.e., $p^* = t(v, b)$ maximizes $\Pi_{\mathrm{t}}(v, p)$. Differentiating a threshold bidder's expected utility function (3) in p, we get

$$\frac{\partial \Pi_{\mathrm{t}}(v, p)}{\partial p} = u(v - b) G'_{n-1}(p) + u(v - p) F^{n-1\prime}(p). \tag{6}$$

Differentiating $G_{n-1}(p)$ gives

$$G'_{n-1}(p) = -(n - 1)(1 - T(p))^{n-2} T'(p) - F^{n-1\prime}(p).$$

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[6] In the interest of simplicity, we assume that t is continuously differentiable in both variables v and b so long as it is above r.

Because $t(v,b)$ is strictly decreasing and continuous in v when $b \leqslant v < \tilde{v}$, its inverse function in the first variable, denoted by $w(\cdot, b)$, exists:

$$t(w(p,b),b) = p.$$

Using w, $T(p)$ can be expressed as

$$T(p) = 1 - F(w(p,b))$$

and we have

$$T'(p) = -\frac{f(w(p,b))}{\partial_1 t(w(p,b),b)}.$$

Replacing $T(p)$ and $T'(p)$ in $G'_{n-1}(p)$, we get

$$\begin{aligned} G'_{n-1}(p) &= (n-1)F^{n-2}(w(p,b))\frac{f(w(p,b))}{\partial_1 t(w(p,b),b)} - F^{n-1'}(p) \\ &= \frac{F^{n-1'}(w(p,b))}{\partial_1 t(w(p,b),b)} - F^{n-1'}(p). \end{aligned}$$

Therefore, we have

$$\frac{\partial \Pi_t(v,p)}{\partial p} = u(v-p)F^{n-1'}(p) + u(v-b)\left(\frac{F^{n-1'}(w(p,b))}{\partial_1 t(w(p,b),b)} - F^{n-1'}(p)\right).$$

For $v > b$, dividing the above equation by $u(v-b)F^{n-1'}(p) > 0$ preserves its sign, and we get

$$\frac{u(v-p)}{u(v-b)} - 1 + \frac{F^{n-1'}(w(p,b))}{\partial_1(F^{n-1} \circ t)(w(p,b),b)}. \tag{7}$$

When t is the equilibrium threshold function, i.e., $p^* = t(v,b)$ and $w(p^*,b) = v$, (7) is zero:

$$\frac{u(v-t(v,b))}{u(v-b)} - 1 + \frac{F^{n-1'}(v)}{\partial_1(F^{n-1} \circ t)(v,b)} = 0.$$

For $v = b$, $t(v,b)$ is continuous in v and $t(b,b) = \lim_{v \searrow b} t(v,b) = b$. By our assumption, t is continuously differentiable. Applying the L'Hospital's rule, we have

$$\lim_{v \searrow b} \frac{u(v-t(v,b))}{u(v-b)} = \frac{u'(0)(1-\partial_1 t(b,b))}{u'(0)} = 1 - \partial_1 t(b,b).$$

We can use this to take the limit in (5) as $v \searrow b$ to obtain $\partial_1 t(b,b) = \pm 1$. Since t is decreasing, $\partial_1 t(b,b) = -1$ must hold. □

Eq. (5) is an ordinary differential equation for $t(\cdot, b)$ with the boundary condition $t(b,b) = b$. The equation always has a unique solution of $t(v,b)$. Although we cannot express the general solution explicitly, bidders in practice can apply (5) to calculate their optimal threshold prices once the characteristics of an auction (e.g., bidders' value distribution, utility functions, and the seller's buy price) become known. Later we will demonstrate the use of (5) when a bidder has constant absolute risk aversion (CARA) utility [2].

Also using (5) we can further verify Proposition 2. Since $u(v-t(v,b)) > u(v-b)$, from (5) we get

$$\frac{F^{n-1'}(v)}{F^{n-1'}(t(v,b))\partial_1 t(v,b)} = 1 - \frac{u(v-t(v,b))}{u(v-b)} < 0$$

which implies $\partial_1 t(v,b) < 0$.

Now we prove that the threshold function defined by (5) is the best response threshold value.

Proposition 5. *Let t be the function defined by* (5) *and $\tilde{v} > b$ satisfy that for all $x < \tilde{v}$ that $t(x,b) > r$. If all other bidders with valuations x, $b \leqslant x < \tilde{v}$, follow the threshold strategy $t(x,b)$, then the optimal threshold strategy of a bidder with valuation v, $b \leqslant v < \tilde{v}$, is to use $t(v,b)$ as his threshold price.*

Proof. To show that $t(v,b) = p^*$ maximizes the expected profit of a bidder with v ($b \leqslant v < \tilde{v}$), it is enough to show that $\frac{\partial \Pi_t(v,p)}{\partial p}$ is positive if $p < t(v,b)$ and is negative when $p > t(v,b)$. From the detailed proof in Appendix C, we get $\Pi_t(v,p)$ strictly increasing for all $p \in (r, t(v,b))$ and strictly decreasing for $p \in (t(v,b), b)$. This proves that, as long as all other bidders use the threshold strategy, $p^* = t(v,b)$ maximizes $\Pi_t(v,p)$ for a given v, $b \leqslant v < \tilde{v}$; that is, the optimal threshold price of a bidder with valuation v is $t(v,b)$. □

Corollary 6. *If $t(v,b)$ defined by* (5) *satisfies*

$$\lim_{v \to \overline{v}} t(v,b) \geqslant r$$

then it is an equilibrium that all bidders with valuation $v \geqslant b$ use the threshold strategy with $t(v,b)$ as their threshold price.

Proof. Proposition 5 with $\tilde{v} = \hat{v} = \overline{v}$ shows that a bidder cannot improve his profit by using a different threshold strategy. We now show that he cannot improve his profit by switching to either the conditional or the unconditional strategy. To do so, it is sufficient to show

$$\Pi_t(v, t(v,b)) > u(v-r)F^{n-1}(r) + u(v-b)(1 - F^{n-1}(r)) > u(v-b).$$

The middle of the inequality above is the bidder's maximum possible profit using the conditional strategy: a conditional bidder reaches his maximum possible expected profit when he pays r if everyone else has a valuation below r and pays b otherwise. It is higher than the maximum profit possible using the unconditional strategy, which is $u(v-b)$.

When all other bidders follow the threshold strategy, by Proposition 5, we have

$$\Pi_t(v, t(v,b)) > \lim_{p \to r} \Pi_t(v,p)$$

and using (3) and $G_{n-1}(p) = F^{n-1}(\tilde{v}) - F^{n-1}(p)$, we have $\lim_{p \to r} \Pi_t(v, p) = u(v - b)(1 - F^{n-1}(r)) + u(v - r)F^{n-1}(r)$. □

Corollary 7. *If* $t(v, b)$ *defined by* (5) *satisfies*

$$\lim_{v \to \overline{v}} t(v, b) \geqslant r$$

then the bidder with the highest valuation wins. When the buyers and seller are risk-neutral, the buyers' and the seller's expected profits are the same as in a standard English auction.

Proof. By Corollary 6, if $\lim_{v \to \overline{v}} t(v, b) \geqslant r$, all bidders follow the threshold strategy with threshold prices strictly decreasing with their valuations. Hence the bidder with the highest valuation reaches his threshold price first and wins the auction. By the revenue equivalence theorem [8,10], a buy-price English auction yields the same expected revenues as a standard English auction when the buyers and the seller are risk-neutral. □

Corollary 6 shows that when $\lim_{v \to \overline{v}} t(v, b) \geqslant r$, it is an equilibrium for all bidders with $v \geqslant b$ to follow the threshold strategy, i.e., $\tilde{v} = \hat{v} = \overline{v}$. Now we need to prove $\tilde{v} < \overline{v}$ when $\lim_{v \to \overline{v}} t(v, b) < r$.

We can calculate the equilibrium $\hat{v}$ and $\tilde{v}$ using (1) and (2). First, find the equation that gives $\hat{v}$ for a given $\tilde{v}$.

One of the following must be true: $\Pi_u(v) \geqslant \Pi_c(v)$ for all $v \geqslant \tilde{v}$ (only use the unconditional strategy), $\Pi_u(v) \leqslant \Pi_c(v)$ for all $v \geqslant \tilde{v}$ (only use the conditional strategy), or there exists $\hat{v}$, $\hat{v} > \tilde{v}$ which satisfies $\Pi_u(\hat{v}) = \Pi_c(\hat{v})$ (use unconditional or conditional strategy, respectively, in different value ranges). The last case leads to

$$\sum_{k=0}^{n-1} F^k(\hat{v})(1 - F^{n-k-1}(\tilde{v})) = n\left(\frac{u(\hat{v} - r)}{u(\hat{v} - b)} - 1\right)F^{n-1}(r). \tag{8}$$

For a given $\tilde{v}$, the right-hand side above is strictly decreasing in $\hat{v}$, while the left-hand side is strictly increasing. Therefore, there is either one unique $\hat{v}$ or no $\hat{v}$ solution ($\hat{v} > \tilde{v}$). If there is no $\hat{v}$ solution, then for all $v \geqslant \tilde{v}$ either $\Pi_u(v) \leqslant \Pi_c(v)$ (i.e., $\hat{v} = \overline{v}$) satisfying

$$\sum_{k=0}^{n-1} (1 - F^{n-k-1}(\tilde{v})) \leqslant n\left(\frac{u(\overline{v} - r)}{u(\overline{v} - b)} - 1\right)F^{n-1}(r) \tag{9}$$

or $\Pi_u(v) \geqslant \Pi_c(v)$ (i.e., $\hat{v} = \tilde{v}$) satisfying

$$\sum_{k=0}^{n-1} F^k(\tilde{v})(1 - F^{n-k-1}(\tilde{v})) \geqslant n\left(\frac{u(\tilde{v} - r)}{u(\tilde{v} - b)} - 1\right)F^{n-1}(r). \tag{10}$$

Define $\Pi_d(v) = \max\{\Pi_c(v), \Pi_u(v)\}$. $\tilde{v}$ is the valuation limit where bidders switch from the threshold strategy either to the conditional or the unconditional strategy; thus, $\Pi_t(\tilde{v}, t(\tilde{v}, b)) = \Pi_d(\tilde{v})$. Since both sides are continuous, in order to demonstrate that

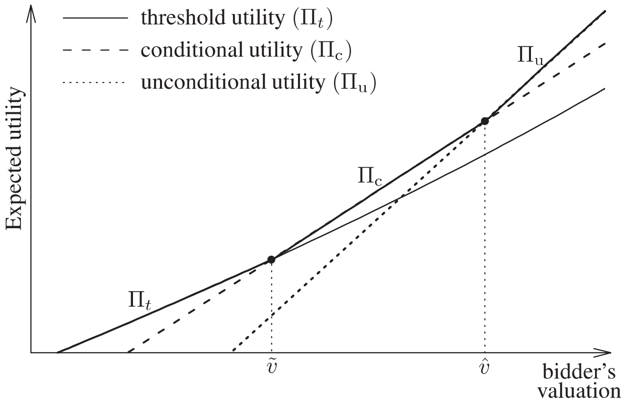

Fig. 2. The relationship among the slopes of the three expected bidder profit functions under the threshold, conditional, and unconditional strategies guarantees the existence of unique $\tilde{v}$ and $\hat{v}$. For simplicity, we depict the functions as linear. In reality, they are non-linear.

there is a solution to this equation, it is sufficient to show that $\Pi_t(\tilde{v}, t(\tilde{v}, b))$ is greater when $\tilde{v} \to b$, but $\Pi_d(\tilde{v})$ is greater when $\tilde{v}$ is large, s.t., $t(\tilde{v}, b) \to r$. First consider the case where $\tilde{v} \to b$:

$$\lim_{\tilde{v} \searrow b} \Pi_u(\tilde{v}) = 0,$$

$$\lim_{\tilde{v} \searrow b} \Pi_c(\tilde{v}) = (b - r)F^{n-1}(r),$$

$$\lim_{\tilde{v} \searrow b} \Pi_t(\tilde{v}, t(\tilde{v}, b)) = \int_r^b u(b - x)\, \mathrm{d}F^{n-1}(x) + (b - r)F^{n-1}(r)$$

which shows that when $\tilde{v} \to b$, $\Pi_t(\tilde{v}, t(\tilde{v}, b)) > \Pi_d(\tilde{v})$ holds.

Let v_x satisfy $t(v_x, b) = r$. Note that $v_x < \overline{v}$ exists here because $\lim_{v \to \overline{v}} t(v, b) < r$. We have

$$\lim_{\tilde{v} \to v_x} \Pi_t(\tilde{v}, t(\tilde{v}, b)) = (u(v_x - r) - u(v_x - b))F^{n-1}(r) + u(v_x - b)F^{n-1}(v_x)$$
$$\leqslant \Pi_c(v_x) \leqslant \Pi_d(v_x).$$

Therefore, there exists a $\tilde{v} \in (b, v_x]$ satisfying $\Pi_t(\tilde{v}, t(\tilde{v}, b)) = \Pi_d(\tilde{v})$ and $\tilde{v} < \overline{v}$.

To show that $\tilde{v}$ and $\hat{v}$, together with t, correspond to an equilibrium, we will use the following inequality (see Fig. 2), which follows easily from Eqs. (1)–(3) and the concavity of u:

$$\Pi_u'(v) \geqslant \Pi_c'(v) \geqslant \frac{\partial \Pi_t(v, p)}{\partial v} \quad \text{for all } v \geqslant b \text{ and } r < p \leqslant b. \tag{11}$$

Intuitively, we can think that a bidder's chance of winning decreases in the order of using the unconditional, conditional, and threshold strategies. The marginal expected profit from the unconditional strategy is the highest followed by the conditional and then the threshold strategies. In addition, a bidder's utility conditional on winning the auction is the smallest in the unconditional strategy, followed by the conditional and the threshold strategies. Thus the above inequality holds.

Inequality (11) implies that for all p, $\Pi_t(v, p) - \Pi_d(v)$ is non-increasing in v. For any $v < \tilde{v}$,

$$\Pi_t(v, t(v, b)) - \Pi_d(v) > \Pi_t(v, t(\tilde{v}, b)) - \Pi_d(v) \geqslant \Pi_t(\tilde{v}, t(\tilde{v}, b)) - \Pi_d(\tilde{v}) = 0$$

which implies that bidders with valuation below $\tilde{v}$ cannot improve their expected utility by switching to the conditional or unconditional strategies. For any $v > \tilde{v}$ and for any p,

$$\Pi_t(v, p) - \Pi_d(v) \leqslant \Pi_t(\tilde{v}, p) - \Pi_d(\tilde{v}) \leqslant \Pi_t(\tilde{v}, t(\tilde{v}, b)) - \Pi_d(\tilde{v}) = 0$$

which implies that bidders with valuation above $\tilde{v}$, following the better of the conditional and unconditional strategies, cannot gain by switching to a threshold strategy.

Inequality (11) also implies that $\Pi_u(v) - \Pi_c(v)$ is non-decreasing. Therefore, if $\Pi_c(\hat{v}) = \Pi_u(\hat{v})$, then for all $v \in [\tilde{v}, \hat{v})$ bidders prefer the conditional strategy and, for all $v \geqslant \hat{v}$, bidders prefer the unconditional strategy. If there is no $\hat{v}$ that satisfies $\Pi_u(\hat{v}) = \Pi_c(\hat{v})$, then either (9) or (10) holds (i.e., one of the two strategies is always strictly better than the other).

Now we have shown that t, $\tilde{v}$, and $\hat{v}$ indeed determine an equilibrium. □

We can prove that $\tilde{v}$ and $\hat{v}$ are unique and the equilibrium described in Theorem 1 is the only pure strategy symmetric equilibrium of a buy-price English auction. The proof is not difficult and uses similar techniques as in the proof of Theorem 1. Since the proof does not provide any more economical insights, we choose to omit it in this paper.

2.3. Threshold prices and the bidders' degrees of risk aversion

Now we examine the relationship between a bidder's threshold price and his absolute level of risk aversion. Assuming that the bidder's valuation is unchanged, the following theorem proves that the more risk-averse a bidder, the lower his threshold price. In other words, the more risk-averse a bidder is, the sooner he would use the buy price in order to avoid the risk that someone else may use it first.

Theorem 8. *Let u_1, u_2 be concave or linear utility functions, t_1, t_2 be the corresponding threshold-price functions, and $a_1 = -u_1''/u_1'$, $a_2 = -u_2''/u_2'$ be the absolute level of risk aversion. If $a_1(x) \leqslant a_2(x)$ for all $x \geqslant 0$, then $t_1(v, b) \geqslant t_2(v, b)$ for all $v \geqslant b$.*

Proof. Prove by contradiction: assume that for all $x \geqslant 0$, $a_1(x) \leqslant a_2(x)$, but there exists $\beta > b$ such that $t_1(\beta, b) < t_2(\beta, b)$. Let $\alpha = \max\{v : v < \beta \wedge t_1(v, b) = t_2(v, b)\}$. α exists because the set over which we take the maximum is closed, bounded from above, and non-empty (e.g., includes b). Then for all v, $\alpha < v \leqslant \beta$, $t_1(v, b) < t_2(v, b)$.

Using Lemma D.1 from Appendix D, the following inequalities hold:

$$\frac{u_1(v - t_1(v, b))}{u_1(v - b)} \geqslant \frac{u_2(v - t_1(v, b))}{u_2(v - b)} > \frac{u_2(v - t_2(v, b))}{u_2(v - b)}.$$

This, combined with (5), implies $\partial_1(F^{n-1} \circ t_1)(v, b) > \partial_1(F^{n-1} \circ t_2)(v, b)$, which means that $F^{n-1}(t_1(v, b)) - F^{n-1}(t_2(v, b))$ is strictly increasing in v for $\alpha < v \leqslant \beta$. Since $F^{n-1}(t_1(\alpha, b)) = F^{n-1}(t_2(\alpha, b))$, $F^{n-1}(t_1(v, b)) > F^{n-1}(t_2(v, b))$ is only possible if $t_1(v, b) > t_2(v, b)$ for all $\alpha < v \leqslant \beta$, which is a contradiction. □

3. The expected social welfare

3.1. Bidders' choice: buy-price or standard English auction?

When a bidder needs to choose between a buy-price and a standard English auction, which one should he prefer? To decide, we need to compare his expected profits. For a bidder with a valuation below b, the two auctions are equivalent because his equilibrium strategy remains the same. For a bidder with valuation at or above b but below $\tilde{v}$, he follows the same strategy as in the standard auction as long as the second-highest bidder valuation is below the threshold price. Otherwise, his expected extra gain from attending a buy-price English auction, instead of a standard one, is

$$\int_{t(v,b)}^{v} (u(v-b) - u(v-x))\,\mathrm{d}F^{n-1}(x). \tag{12}$$

Next, we calculate the bidder's gains when he is risk-averse or neutral, respectively. Suppose the bidder has CARA utility $u_a(x) = (1-e^{-ax})/a$, where $a > 0$ is the absolute level of risk aversion. If the bidder is risk-neutral, $a = 0$, $u_0(x) = \lim_{a\to 0} u_a(x) = x$.[7] The CARA utility satisfies the following:

$$\frac{u_a(x) - u_a(y)}{u_a'(y)} = u_a(x-y) \quad \text{for all } a \geqslant 0,\, x, y \in \mathbb{R},$$

$$u_a(-x) = -\frac{u_a(x)}{u_a'(x)} \quad \text{for all } a \geqslant 0,\, x \in \mathbb{R}.$$

Applying them in (5), we get

$$u_a(b - t(v,b))\left(F^{n-1} \circ t(\cdot, b)\right)'(v) = u_a(b-v)F^{n-1'}(v).$$

Solving it with the boundary condition $t(b,b) = b$ yields

$$\int_{b}^{t(v,b)} u_a(b-x)\,\mathrm{d}F^{n-1}(x) = \int_{b}^{v} u_a(b-x)\,\mathrm{d}F^{n-1}(x),$$

$$\int_{t(v,b)}^{v} u_a(b-x)\,\mathrm{d}F^{n-1}(x) = 0. \tag{13}$$

Using Eq. (13), a bidder with CARA utility can calculate his threshold price $t(v,b)$.

Rewriting (12) and using (13), we get

$$\int_{t(v,b)}^{v} (u_a(v-b) - u_a(v-x))\,\mathrm{d}F^{n-1}(x) = -u_a'(v-b)\int_{t(v,b)}^{v} u_a(b-x)\,\mathrm{d}F^{n-1}(x) = 0.$$

Hence, a bidder with CARA utility and a valuation v, $b \leqslant v < \tilde{v}$ gains no extra expected profit from attending a buy-price English auction instead of a standard one.

[7] Subsequently, whenever we mention CARA utility we also include the linear utility function.

However, bidders with valuation above $\tilde{v}$ are no longer indifferent between a buy-price and a standard English auction. Most bidders will be better off in a buy-price English auction. If the buy price is low and $\hat{v} < \overline{v}$, however, then some bidders with very high valuations will prefer the standard English auction where they do not have to participate in the random draw with other unconditional bidders; thus, their chance of winning is higher. We have not calculated the exact conditions under which a bidder with a high valuation prefers a standard English auction, but we conjecture that this could not happen with most value distributions unless the buy price is set to unreasonably low levels.

3.2. *Seller's choice: buy-price or standard English auction?*

3.2.1. *Risk-neutral sellers*

Define $t_{\overline{v}} = \lim_{v \to \overline{v}} t(v, b)$. We have seen that when $t_{\overline{v}} \geqslant r$ is in equilibrium, all bidders follow the threshold strategy with a strictly decreasing t, ensuring that the bidder with the highest valuation wins. The revenue equivalence theorem, which is true only when both the seller and the bidders are risk-neutral, implies that a risk-neutral seller's expected profit in a buy-price auction is the same as that in a standard one. On the other hand, when $t_{\overline{v}} \geqslant r$ does not hold, bidders with valuations above $\tilde{v}$ follow different strategies and the auction no longer guarantees that the bidder with the highest valuation wins.

When bidders have CARA utility and $t_{\overline{v}} \geqslant r$, we can derive the following from Eq. (13):

$$0 = \int_{t_{\overline{v}}}^{\overline{v}} u_a(b - x)\, \mathrm{d}F^{n-1}(x) \leqslant \int_{r}^{\overline{v}} u_a(b - x)\, \mathrm{d}F^{n-1}(x)$$

which implies

$$\lim_{v \to \overline{v}} t(v, b) = t_{\overline{v}} \geqslant r \iff \int_{r}^{\overline{v}} u_a(b - x)\, \mathrm{d}F^{n-1}(x) \geqslant 0. \qquad (14)$$

When buyers are risk-neutral, i.e., $a = 0$ and $u_a(b - x) = b - x$, the condition of (14) is equivalent to

$$\int_{r}^{\overline{v}} b\, \mathrm{d}F^{n-1}(x) \geqslant \int_{r}^{\overline{v}} x\, \mathrm{d}F^{n-1}(x)$$

which can be rewritten as

$$b \geqslant \int_{r}^{\overline{v}} x\, \frac{\mathrm{d}F^{n-1}(x)}{1 - F^{n-1}(r)}. \qquad (15)$$

This result can be summarized in the following theorem (see Fig. 3):

Theorem 9. *In a buy-price English auction, if both the seller and buyers are risk-neutral and the buy price is set at least as high as the expected maximum valuation among* $(n - 1)$ *buyers on the condition that at least one of the* $(n - 1)$ *buyers has a valuation at or above the reserve price, then the seller's expected profit is the same as that in a standard English auction, and the buy-price English auction is efficient.*

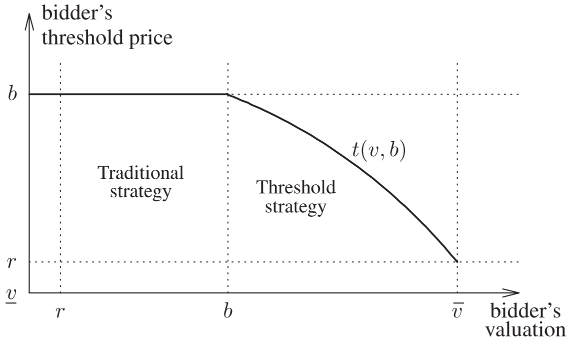

Fig. 3. A buy-price English auction is efficient if $b \geqslant \int_r^{\overline{v}} x \, \frac{\mathrm{d}F^{n-1}(x)}{1-F^{n-1}(r)}$.

If the second-highest bidder valuation is below the reserve price, the winning bidder would only pay the reserve in both the standard and buy-price auctions. Therefore, to compare the two auctions, it is sufficient to consider the expected seller revenues conditional on having at least two bidders with valuations no lower than the reserve. For the following discussion we assume this condition.

How high should a buy price be? We know that the buy price is the maximum revenue the seller gets from a buy-price auction, thus the expected seller revenue in such an auction is always less than the buy price. This implies that for the seller to have the same expected revenue from the buy-price and standard auctions, she has to set the buy price higher than her expected revenue from a standard auction (i.e., the expected second-highest bidder valuation).

The criterion on how to choose a good buy price in Theorem 9 follows the above intuition. The maximum valuation of arbitrary $(n-1)$ bidders is usually higher than the second-highest valuation. They are equal only when the chosen $(n-1)$ bidders happen to be the bidders with the lowest $(n-1)$ valuations, but in all other cases the maximum valuation of arbitrary $(n-1)$ bidders is equivalent to the maximum valuation among all bidders.

Another intuitive way to obtain a lower bound of a well-chosen buy price that guarantees the revenue equivalence between buy-price and standard auctions is to study the expected payment of the bidder with the highest type $\overline{v}$. In the standard auction, the $\overline{v}$-type bidder's expected payment is the expected maximum valuation of all other bidders, which, if the revenue equivalence holds, is also his expected payment in the buy-price auction. Since his maximum payment is the buy price, the buy price must be at least as high as the expected maximum valuation of all other bidders, which is exactly what is depicted in inequality (15).

Inequality (15) is important because it helps the seller to choose an appropriate buy price. Example 10 demonstrates how to calculate the lower bound for the buy price.

Example 10. In an auction where there are two risk-neutral bidders with valuation drawn from the uniform [0, 1] distribution and the seller's valuation is 0, the optimal reserve price is 0.5 and the lowest buy price that satisfies the revenue equivalence is 0.75.

In this example, $F(x) = x$ and $f(x) = 1$ for $x \in [0, 1]$. The optimal reserve price satisfies $r = (1 - F(r))/f(r) = 1 - r$, thus $r = 0.5$. From inequality (15) we get

$$b \geqslant \int_r^{\overline{v}} x \, \frac{\mathrm{d}F^{n-1}(x)}{1 - F^{n-1}(r)} = \int_{0.5}^{1} x \, \frac{\mathrm{d}x}{1 - 0.5} = 1^2 - 0.5^2 = 0.75.$$

Note that this lower bound of the buy price only applies to auctions with two bidders. As the number of bidders increases, the buy price should also increase in order to ensure the revenue equivalence.

The same lower bound can also derived from (13):

$$\begin{aligned} 0 &= \int_{t(v,b)}^{v} u_a(b - x) \, \mathrm{d}F^{n-1}(x) \\ &= \int_{t(v,b)}^{v} (b - x) \, \mathrm{d}x = (v - t(v, b)) \frac{2b - v - t(v, b)}{2} \end{aligned}$$

which implies that the threshold function $t(v, b) = 2b - v$. v can be at most 1, so the condition for the threshold function staying above the reserve is $t(1, b) = 2b - 1 \geqslant r = 0.5 \Longrightarrow b \geqslant 0.75$. 0.75 is also the expected maximum valuation of one (i.e., $n - 1$) bidder conditional on his valuation (the valuation of one out of $n - 1 = 1$) is at least 0.5. Such a bidder has a uniform [0.5, 1] valuation distribution with an expected valuation of 0.75.

We can also characterize the expected profit of a risk-neutral seller facing risk-averse buyers in the following theorem:

Theorem 11. *In a buy-price English auction, if the seller is risk-neutral, the buyers are risk-averse, and the buy price is set at least as high as the expected maximum valuation among* $(n - 1)$ *buyers on the condition that at least one of the* $(n - 1)$ *buyers has a valuation at or above the reserve price, then the seller's expected profit is higher than that in a standard English auction.*

Proof. Theorem 8 says that the more risk-averse the buyers are, the lower their threshold prices. This increases the seller's expected profit because more buyers will pay the buy price instead of the second-highest bidder valuation. But in a standard auction, the seller's expected profit does not change with the buyers' levels of risk aversion, as buyers bid up to their valuations regardless. Therefore, when buyers are risk-averse, a risk-neutral seller is better off in a buy-price English auction. □

3.2.2. Risk-averse sellers

Let us now calculate the expected profit of a risk-averse seller with risk-neutral buyers. The calculation presented below is similar to that of Riley and Samuelson [10] except that the seller's utility function $s(x)$, where x denotes the sale price, is more general because the seller under analysis can be either risk-neutral or risk-averse: $s(x) = x$ if the seller is risk-neutral and $s(x)$ is strictly concave if the seller is risk-averse.

At the equilibrium, the seller's expected profit from a bidder with a valuation below b is the same as in a standard auction. It is given by Eq. (8b) in [10]

$$\begin{aligned} P_0(v) &= s(r)F^{n-1}(r) + \int_r^v s(x)\,\mathrm{d}F^{n-1}(x) \\ &= s(v)F^{n-1}(v) - \int_r^v s'(x)F^{n-1}(x)\,\mathrm{d}x. \end{aligned}$$

The seller's expected profit from a bidder with a valuation $v \in [b, \tilde{v})$ is

$$P(v,b) = P_0(t) + s(b)(F^{n-1}(v) - F^{n-1}(t)).$$

Hence, the seller's overall expected profit from all n bidders, denoted by $\Pi_{\mathrm{s}}(v,b)$, is:

$$\begin{aligned} \Pi_{\mathrm{s}}(v,b) = n &\left(\int_r^b P_0(v)\,\mathrm{d}F(v) + \int_r^{\tilde{v}} P(v,b)\,\mathrm{d}F(v) \right) \\ &+ b(1 - F^n(\tilde{v})) - n(b-r)F^{n-1}(r)(F(\hat{v}) - F(\tilde{v})). \end{aligned} \tag{16}$$

Since we can regard no buy price in a standard auction as having a very large buy price, i.e., $b \to \infty$, proving that a risk-averse seller is better off in a buy-price English auction with risk-neutral buyers is equivalent to showing that the b-derivative of $\Pi_{\mathrm{s}}(v,b)$ is negative when (15) holds. The b-derivative of $\Pi_{\mathrm{s}}(v,b)$, when $\tilde{v} = \hat{v} = \overline{v}$, is

$$\begin{aligned} \partial_2 \Pi_{\mathrm{s}}(v,b) &= n \int_b^{\overline{v}} \partial_2 P(v,b)\,\mathrm{d}F(v) \\ &= n \int_b^{\overline{v}} \Big[(s(t) - s(b))\partial_2(F^{n-1} \circ t)(v,b) \\ &\qquad + s'(b)(F^{n-1}(v) - F^{n-1}(t)) \Big]\,\mathrm{d}F(v). \end{aligned}$$

Differentiating (13) by b with $a = 0$, i.e., $u_a(x) = x$, we get

$$F^{n-1}(v) - F^{n-1}(t) = (b-t)\left(\partial_2(F^{n-1} \circ t)\right)(v,b). \tag{17}$$

Applying (17) and $s(b) - s(t) \geqslant s'(b)(b-t)$ due to the concavity of s, we get

$$\partial_2 \Pi_{\mathrm{s}}(v,b) \leqslant 0.$$

The inequality shows that as long as (15) holds—i.e., the buy price is set high enough that no one uses a conditional or unconditional strategy—a risk-averse seller is strictly better off in a buy-price English auction than in a standard one when buyers are risk-neutral. The equality holds if and only if the seller is risk-neutral, implying that the seller's expected profits from a buy price is the same as in a standard auction. Moreover, risk-averse buyers further increase the seller's expected profit. Hence, the following theorem holds:

Theorem 12. *In a buy-price English auction, if either the seller or the buyers are risk-averse and the buy price is set at least as high as the expected maximum valuation among* $(n-1)$ *buyers on the condition that at least one of the* $(n-1)$ *buyers has a valuation at or* *above the reserve price, then the seller's expected profit is higher than in a standard English auction.*

Therefore, we can conclude that regardless of whether the seller is risk-neutral or risk-averse, she cannot lose by utilizing buy prices in English auctions.

4. Concluding remarks

This paper has analyzed buy-price English auctions of an indivisible good in the general setting of n bidders with continuous, independently distributed, and private valuations. We have proved that in equilibrium unique reference points $\tilde{v}$ and $\hat{v}$ exist ($\tilde{v} \leqslant \hat{v}$, with both above the buy price) so that a bidder with a valuation between the buy price and $\tilde{v}$ bids the buy price when the current high bid reaches a threshold price (i.e., the competition among bidders is heated and has reached a level which makes such a bidder unwilling to risk waiting further and thus bids the buy price), a bidder with a valuation between $\tilde{v}$ and $\hat{v}$ bids the buy price on the condition that there already exists a valid bid above or equal to the reserve (i.e., at least one competing bidder exists), and a bidder with a valuation at or above $\hat{v}$ bids the buy price unconditionally (i.e., regardless of whether there is competition or not).

We have proved that the threshold bidders' equilibrium threshold prices can be calculated using the buy price, bidders' utility functions, and their value distribution. These threshold prices are between the reserve and the buy prices and strictly decreasing with valuations. In other words, the higher a threshold bidder's valuation, the lower his threshold price.

We have also shown that if the buy price is set at or above a lower bound, then all bidders with valuations above the buy price are threshold bidders; that is, there are no conditional or unconditional bidders. Since the threshold prices are strictly decreasing with the bidders' valuations, the bidder with the highest valuation will reach his threshold price first, and thus the auction guarantees that the highest bidder wins and the equilibrium yields full efficiency. In addition, the more risk-averse a threshold bidder, the lower his threshold price. In other words, a more risk-averse bidder tends to bid the buy price earlier, which helps to explain why the seller can gain higher expected profit from risk-averse buyers than from risk-neutral buyers.

Clearly, the buy-price option can reduce a buyer's risk: bidding the buy price, a buyer can obtain the item at a fixed price, and he thus no longer has to worry about losing the auction to a bidder with a higher valuation. Because of this observation one may expect that risk-averse buyers are better off in buy-price English auctions, but this is not true. To reduce the risk of losing the auction, risk-averse buyers bid the buy price more often than risk-neutral bidders. They may bid the buy price even in cases where there are no other bidders with valuation above the buy price, and therefore, pay more than they would have in a standard English auction. In fact, we have proved that when buyers are risk-neutral or uniformly risk-averse, and when the buy price is properly set above a lower bound, the buyers' expected utility in a buy-price English auction is the same as in a standard one.

If risk-averse buyers' expected utility does not increase with the introduction of a buy price even though their risks are reduced, then their expected payment must increase to offset the positive utility of the reduced risk. This insight again explains why the seller's expected revenue is higher in a buy-price English auction than in a standard one when bidders are risk-averse.

In addition to the seller's higher or equivalent expected revenue, the seller's risk is also reduced in a buy-price English auction because the seller will often get the buy price instead of some unpredictable payment either below or above the buy price. This observation in turn implies that a risk-averse seller always prefers a buy-price English auction because in it her expected revenue is not lower than that in a standard one when the buy price is properly set and, at the same time, her risk is reduced.

The above results do not follow in cases with temporary and limited buy prices, as neither auction can guarantee that the bidder with the highest valuation wins when all agents are risk-neutral. In an English auction with a temporary buy price, bidders have to decide whether or not to use the buy price without observing any bid, and, therefore, their best symmetric strategy is to find a valuation level above which they unconditionally exercise the buy price. This leads to inefficient outcomes reducing the seller's expected profit [1]. Similarly, an English auction with a limited buy price is also inefficient. Although the temporary and limited buy prices can increase the social welfare when players are risk-averse [7,9], they lower the expected social welfare when players are risk-neutral. Thus temporary and limited buy-price English auctions are generally inferior to the ones with permanent buy prices. This result implies that the permanent buy-price auctions offered by Yahoo!, uBid, and Amazon are, in theory, more beneficial to all agents than the temporary buy-price auctions, like those offered by eBay, or the limited buy-price auctions, like those offered by labx, especially for unique and used goods where buyers have private valuations and face relatively high risks. While the positive network externality has contributed significantly to the popularity of eBay, features in Yahoo!, uBid, and Amazon auctions also have their own competitive advantages. [8] For practice, we recommend that auction houses choose appropriate policies with respect to buy prices, conduct market research on the players' degrees of risk aversion in different markets, and suggest strategies to auction sellers how to use buy prices for additional revenue.

Relaxing the assumptions of the revenue equivalence theorem [8,10] leads to differences among the English, Dutch, first-price sealed-bid, and second-price sealed-bid auction mechanisms [12]. Maskin and Riley [5] provide a detailed analysis of auctions with risk-averse buyers. The most notable result related to our research is that when bidders are risk-averse, first-price sealed-bid and Dutch auctions provide higher expected seller profit than second-price auctions. Using a two-bidder two-type model, Budish and Takeyama [3] conclude that a buy-price English auction can be superior even to the first-price sealed-bid and Dutch auctions when bidders are risk-averse. It would be interesting to investigate whether this result remains true in a general setting of n bidder with arbitrary value distribution.

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[8] In addition to the good policy of utilizing buy prices, auctions in Yahoo!, uBid, and Amazon have some other advanced features. For instance, at the time when this paper is written, Yahoo! and uBid authenticate buyers more rigorously than eBay by requiring valid credit card information for registration. Yahoo! even asks for two passwords for the purpose of authentication, which reduces the number of non-paying winners and the fraud of shill bidding [14]. Yahoo! also allows a seller to specify the automatic extension of an auction if a bid is made within the last 5 min of the auction, and this helps to prevent last-minute bidding [11]. Moreover, Yahoo! auctions charge relatively low intermediation fees.

Another extension of our model would consider the time factor. With the pace of transactions getting faster and the Internet's around-the-clock operations allowing random arrival of traders, the temporal property of a trade becomes increasingly important. We suspect that delay-averse auction sellers and buyers are more likely to use buy prices than delay-neutral or delay-taking buyers, and that the shortened auction cycles would increase the market liquidity. Lucking-Reiley [4] mentioned that the use of a buy price is to "allow buyers to buy an early end to the auction by submitting a sufficiently high bid." Mathews [6] modeled eBay's temporary buy price auction with a time discount and showed that when the seller and buyers are risk-neutral, even temporary buy prices that are exercised with positive probabilities are welcome, because buyers are willing to pay more to get the item sooner and/or the seller is willing to give up some of her expected profit to get the payment sooner. In contrast to an analysis of eBay's temporary buy price model with uniform bidder distribution, we need a more general model to analyze the temporal effect of utilizing permanent buy prices in auctions with arbitrary bidder distribution.

Entry costs to auctions may also affect the comparison between buy-price and standard English auctions. Rothkopf and Harstad [12] note that when potential buyers expect strong competition for an item, they may not invest efforts to enter the auction because the winner can only expect small profits. A buy price can guarantee a minimum profit for the winner and, hence, may attract more bidders.

Another research direction is to study how the seller uses buy prices as signaling devices. Too high a buy price may alienate buyers from bidding. Too low a buy price may convey information on adverse quality. Ultimately, it might even be possible to embed a Dutch auction within an English auction by allowing buy prices to decrease during the auction.

While we have only modeled private value auctions, buy prices could also prove beneficial in common value models. In a common value auction, the buy price is a strong signal from the seller about the value of the item, which can help reduce the errors in the bidders' value estimates and thus may lower the "winner's curse" effect, in turn increasing the seller's expected profit.

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Appendix A. Proof of Proposition 2

Proposition 2. *$t(v, b)$ is strictly decreasing in v when $b \leqslant v < \tilde{v}$.*

Proof. We first prove that $t(v, b)$ is non-increasing in v for all $v \geqslant b$. Prove by contradiction: assume $t(v, b)$ is increasing in v for all $v \geqslant b$, that is, for some $v_0 < v_1$, $t_0 = t(v_0, b) < t_1 = t(v_1, b)$. Since we assume that F is strictly increasing, every bidder would have a unique valuation and there will not be random-draw cases. When the auction clock reaches t_0, a bidder with a valuation v_1 can either jump to the buy price immediately for a guaranteed $u(v_1 - b)$ profit, or continue bidding and wait to jump at t_1. Since by our assumption t_1 is the equilibrium threshold price of this bidder, jumping at t_1 should give him expected profit no less than jumping at t_0:

$$u(v_1 - b)G_{n-1}(t_1) + \int_{t_0}^{t_1} u(v_1 - x)\,\mathrm{d}F^{n-1}(x) \geqslant u(v_1 - b)G_{n-1}(t_0).$$

$$\int_{t_0}^{t_1} u(v_1 - x)\,\frac{\mathrm{d}F^{n-1}(x)}{G_{n-1}(t_0)} \geqslant u(v_1 - b)\left(1 - \frac{G_{n-1}(t_1)}{G_{n-1}(t_0)}\right). \tag{A.1}$$

On the other hand, for the buyer with valuation v_0, jumping to the buy price at t_0 is at least as good as continuing bidding and jumping at t_1:

$$u(v_0 - b)G_{n-1}(t_0) \geqslant u(v_0 - b)G_{n-1}(t_1) + \int_{t_0}^{t_1} u(v_0 - x)\,\mathrm{d}F^{n-1}(x).$$

$$u(v_0 - b)\left(1 - \frac{G_{n-1}(t_1)}{G_{n-1}(t_0)}\right) \geqslant \int_{t_0}^{t_1} u(v_0 - x)\,\frac{\mathrm{d}F^{n-1}(x)}{G_{n-1}(t_0)}. \tag{A.2}$$

Since u is concave, $x \leqslant b \Longrightarrow u(v_1 - x) - u(v_0 - x) \leqslant u(v_1 - b) - u(v_0 - b)$. Together with (A.1), (A.2), we have

$$\int_{t_0}^{t_1} \frac{\mathrm{d}F^{n-1}(x)}{G_{n-1}(t_0)} \geqslant 1 - \frac{G_{n-1}(t_1)}{G_{n-1}(t_0)},$$

$$G_{n-1}(t_0) - G_{n-1}(t_1) \leqslant F^{n-1}(t_1) - F^{n-1}(t_0).$$

Recall that $G_{n-1}(t)$ is defined to be the probability that a bidder with threshold t exercises the buy price and wins the auction. Lowering the threshold from t_1 to t_0 increases the chance of successfully using the buy price by at least the probability that there is another bidder with a valuation between t_0 and t_1, i.e.,

$$G_{n-1}(t_0) - G_{n-1}(t_1) \geqslant F^{n-1}(t_1) - F^{n-1}(t_0).$$

Hence $G_{n-1}(t_0) - G_{n-1}(t_1) = F^{n-1}(t_1) - F^{n-1}(t_0)$, and (A.2) becomes

$$u(v_0 - b)(F^{n-1}(t_1) - F^{n-1}(t_0)) \geqslant \int_{t_0}^{t_1} u(v_0 - x)\,\mathrm{d}F^{n-1}(x),$$

$$0 \geqslant \int_{t_0}^{t_1} (u(v_0 - x) - u(v_0 - b))\,\mathrm{d}F^{n-1}(x)$$

which implies $F(t_0) = F(t_1)$. Since F is strictly increasing, $t_0 = t_1$ must hold, contradictory to our assumption. Therefore, we have proved that $t(v, b)$ is non-increasing in v for all $v \geqslant b$.

Now we further prove that $t(v, b)$ is strictly decreasing when $b \leqslant v < \tilde{v}$. Since by definition $t(\tilde{v}, b) = r$, proving t as strictly decreasing indicates $t(v, b) > r$ when $b \leqslant v < \tilde{v}$. Again, prove by contradiction: assume that for $t_0 > r$ and some $b \leqslant v_0 < v_1, \forall v(v_0 \leqslant v \leqslant v_1 \Longrightarrow t(v, b) = t_0)$, that is, all bidders with valuations between v_0 and v_1 pool at the same threshold price t_0. This implies a positive probability that more than one bidder with valuations between v_0 and v_1 would jump to the buy price simultaneously when t_0 is reached and the winner would be chosen randomly among them. But if one of them decides to jump earlier (i.e., when the auction clock reaches $t_0 - \varepsilon$ instead of t_0 for some arbitrarily small ε), he can avoid this random-draw gamble and increase his chance of winning. By doing so, he can increase his expected profit and only suffer at most ε additional loss. Clearly, if ε is small enough, he can be better off by jumping earlier. Therefore, pooling cannot be an equilibrium. Thus, $t(v, b)$ is strictly decreasing when $b \leqslant v < \tilde{v}$, i.e., $t(v, b) > r$. □

Appendix B. Proof of Proposition 3

Proposition 3. *$t(v, b)$ is continuous in v when $b \leqslant v < \tilde{v}$.*

Proof. We first prove $t(v, b)$ is right-continuous in v when $b \leqslant v < \tilde{v}$, i.e., $t(v, b) > r$. Let $v_0 \geqslant b$, $t_0 = t(v_0, b)$, $v > v_0$, and $t_+ = \lim_{v \searrow v_0} t(v, b)$. The monotonicity of $t(v, b)$ implies $t_+ \leqslant t_0$. Since $t(v, b)$ is the equilibrium threshold, for the buyer with a valuation v, jumping to the buy price when the auction clock reaches $t(v, b) \leqslant t_0$ is at least as good as jumping at t_0:

$$u(v - b)G_{n-1}(t(v, b)) \geqslant u(v - b)G_{n-1}(t_0) + \int_{t(v,b)}^{t_0} u(v - x)\,\mathrm{d}F^{n-1}(x). \qquad \text{(B.1)}$$

Recall that G is continuous. Therefore, we can take the limit in (B.1) as $v \searrow v_0$ to obtain

$$u(v_0 - b) \geqslant u(v_0 - b)\,\frac{G_{n-1}(t_0)}{G_{n-1}(t_+)} + \int_{t_+}^{t_0} u(v_0 - x)\,\frac{\mathrm{d}F^{n-1}(x)}{G_{n-1}(t_+)}.$$

Since $t(v, b)$ is non-increasing in v and now $v \searrow v_0$, $t(v, b)$ cannot take any value between t_+ and t_0. $t(v, b)$ is strictly decreasing when it is above r, thus when $t_+ > r$ we have $T(t_+) = T(t_0)$, which by Eq. (4) implies $G_{n-1}(t_0) - G_{n-1}(t_+) = -(F^{n-1}(t_0) - F^{n-1}(t_+))$. Therefore,

$$0 \geqslant \int_{t_+}^{t_0} u(v_0 - x)\,\frac{\mathrm{d}F^{n-1}(x)}{G_{n-1}(t_+)} - u(v_0 - b)\,\frac{F^{n-1}(t_0) - F^{n-1}(t_+)}{G_{n-1}(t_+)},$$

$$0 \geqslant \int_{t_+}^{t_0} (u(v_0 - x) - u(v_0 - b))\,\frac{\mathrm{d}F^{n-1}(x)}{G_{n-1}(t_+)}.$$

For $x < b$, $u(v_0 - x) - u(v_0 - b) > 0$ and $t_0 \leqslant b$, therefore if $t_0 > t_+$, the integral on the right side would be strictly positive. Therefore, the above formula can only hold if $t_+ = t_0$. This completes the proof of the right-continuity of $t(v, b)$ with regard to v when $t(v, b) > r$. Using similar arguments, we can prove the left-continuity of t. □

Note that the continuity of $t(v, b)$ in v is only true when $t(v, b) > r$; it is possible that t has discontinuity at some point where t suddenly drops to r.

Appendix C. Proof of Proposition 5

Proposition 5. *Let t be the function defined by* (5) *and let $\tilde{v} > b$ satisfy that for all $x < \tilde{v}$ that $t(x, b) > r$. If all other bidders with valuations x, $b \leqslant x < \tilde{v}$, follow the threshold strategy $t(x, b)$, then the optimal threshold strategy of a bidder with valuation v, $b \leqslant v < \tilde{v}$, is to use $t(v, b)$ as his threshold price.*

For the proof we will need the following lemma:

Lemma C.1. *For any $x > 0$ and $d > 0$, $\dfrac{u(x+d)}{u(x)}$ is strictly decreasing in x.*

Proof. Let $x < y$.

$$\begin{aligned} \frac{u(x+d)}{u(x)} - 1 = \frac{u(x+d)-u(x)}{u(x)} &\geqslant \frac{u(y+d)-u(y)}{u(x)} \quad \text{(because } u \text{ is concave)} \\ &> \frac{u(y+d)-u(y)}{u(y)} \quad \text{(because } u \text{ is increasing)} \\ &= \frac{u(y+d)}{u(y)} - 1. \qquad \square \end{aligned}$$

Proof. To show that $t(v, b) = p^*$ maximizes the expected profit of a bidder with $b \leqslant v < \tilde{v}$, it is enough to show that (7), having the same sign as $\frac{\partial \Pi_t(v,p)}{\partial p}$, is positive if $p < t(v, b)$ and negative if $p > t(v, b)$. Since $w(\cdot, b)$ is the inverse of $t(\cdot, b)$, $w(t(v, b), b) = v$. $w(\cdot, b)$ is strictly decreasing; therefore, $w(p, b) < v$ if $p > t(v, b)$ and $w(p, b) > v$ if $p < t(v, b)$.

First consider the case when p is in the range of t, i.e., $v = w(p, b) < \tilde{v}$ and $t(w(p, b), b) = p$. Substitute $v = w(p, b)$ into (5)

$$\begin{aligned} \frac{F^{n-1'}(w(p,b))}{\partial_1(F^{n-1} \circ t)(w(p,b),b)} &= 1 - \frac{u(w(p,b) - t(w(p,b),b))}{u(w(p,b)-b)} \\ &= 1 - \frac{u(w(p,b)-p)}{u(w(p,b)-b)}. \end{aligned}$$

Substituting this into (7) we get

$$\frac{u(v-p)}{u(v-b)} - \frac{u(w(p,b)-p)}{u(w(p,b)-b)}. \tag{C.1}$$

Applying Lemma C.1 with $d = b - p$, $x_1 = v - b$, and $x_2 = w(p, b) - b$, we can see that (C.1) is positive if $w(p, b) > v$, i.e., $p < t(v, b)$, and negative if $w(p, b) < v$, i.e., $p > t(v, b)$. This shows that $t(v, b)$ maximizes the expected profit from the threshold strategy as long as the threshold is in the range of t. Since t is continuously decreasing and $t(b, b) = b$, the range of $t(v, b)$ for $b \leqslant v < \tilde{v}$ is an interval $(\underline{t}, b)$.

When p is not in the range of t, i.e., $r < p \leqslant \underline{t}$ and no other bidder will use a threshold price below or equal to p, using the threshold price p a bidder can only lose to conditional or unconditional bidders. Therefore,

$$G_{n-1}(p) = F^{n-1}(\tilde{v}) - F^{n-1}(p).$$

Applying it in (6) we get

$$\frac{\partial \Pi_t(v, p)}{\partial p} = (u(v-p) - u(v-b))F^{n-1'}(p) > 0$$

which implies that $\Pi_t(v, p)$ is strictly increasing for all $p \in (r, \underline{t}]$.

Combining this with the case in which $p \in [\underline{t}, b)$, we get $\Pi_t(v, p)$ strictly increasing for all $p \in (r, t(v, b))$ and strictly decreasing for $p \in (t(v, b), b)$. This proves that, as long as all other bidders use the threshold strategy, $p^* = t(v, b)$ maximizes $\Pi_t(v, p)$ for a given v; that is, the optimal threshold price of a bidder with valuation v is $t(v, b)$. □

Appendix D. Comparing utility functions based on the level of risk aversion

Lemma D.1. *Let $u_1 : \mathbb{R}^+ \mapsto \mathbb{R}^+$, $u_2 : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be twice differentiable utility functions, $u_1(0) = u_2(0) = 0$, and $u_1'(x) > 0$, $u_2'(x) > 0$ for all $x \geqslant 0$. Let $a_1 = -u_1''/u_1'$ and $a_2 = -u_2''/u_2'$ be the absolute level of risk aversion. If $a_1(x) \leqslant a_2(x)$ for all $x \geqslant 0$, then the following inequality holds*:

$$\forall x, y \left(0 < y < x \Longrightarrow \frac{u_1(x)}{u_1(y)} \geqslant \frac{u_2(x)}{u_2(y)}\right).$$

When $\exists y\ (0 < y < x)$ such that the equality holds, there is a constant λ such that $u_1(z) = \lambda u_2(z)$ for all $0 \leqslant z \leqslant x$.

Proof. Let $\lambda = u_1(x)/u_2(x)$. Part I: We want to prove that $\forall x, y, 0 < y < x$

$$\frac{u_1(x)}{u_2(x)} \geqslant \frac{u_1(y)}{u_2(y)} \iff \lambda \geqslant \frac{u_1(y)}{u_2(y)} \iff \lambda u_2(y) - u_1(y) \geqslant 0.$$

Prove by contradiction: Suppose $\exists y\ (0 < y < x \wedge \lambda u_2(y) - u_1(y) < 0)$, that is, $\exists y \left(0 < y < x \wedge \int_0^y (\lambda u_2'(v) - u_1'(v))\,\mathrm{d}v < 0\right)$. Together with $u_1(0) = u_2(0) = 0$, it implies that there is a z, $0 < z < y$, s.t., $\lambda u_2'(z) - u_1'(z) < 0$, that is, $u_1'(z)/u_2'(z) > \lambda$.

Note that $a_1 = -u_1''/u_1' = (-\ln u_1')'$ and $a_2 = -u_2''/u_2' = (-\ln u_2')'$. Thus for all $v \geqslant 0$

$$a_2(v) - a_1(v) = \left(\ln \frac{u_1'(v)}{u_2'(v)}\right)' \geqslant 0.$$

This means that u_1'/u_2' is non-decreasing. Therefore, $\forall v \left(v \geqslant z \Longrightarrow u_1'(v)/u_2'(v) > \lambda \Longrightarrow \lambda u_2'(v) - u_1'(v) < 0\right)$. Hence,

$$\lambda u_2(x) - u_1(x) = \lambda u_2(y) - u_1(y) + \int_y^x (\lambda u_2'(v) - u_1'(v))\,\mathrm{d}v < 0$$

which contradicts the definition of $\lambda = u_1(x)/u_2(x)$. Therefore, the following must hold:

$$\forall x, y \left(0 < y < x \Longrightarrow \frac{u_1(x)}{u_1(y)} \geqslant \frac{u_2(x)}{u_2(y)} \right).$$

Part II: Assume $\exists y\ (0 < y < x \ \wedge\ \lambda u_2(y) - u_1(y) = 0)$, that is

$$\int_0^y (\lambda u_2'(v) - u_1'(v))\,\mathrm{d}v = 0.$$

Note that u_1'/u_2' is non-decreasing. In order to satisfy the above, either $\forall z\ \big(0 < z < y \Longrightarrow \lambda u_2'(z) - u_1'(z) = 0\big)$ or $\exists z_1, z_2\ \big(0 < z_1 < z_2 < y \ \wedge\ \lambda u_2'(z_1) - u_1'(z_1) > 0 \ \wedge\ \lambda u_2'(z_2) - u_1'(z_2) < 0\big)$. But the later case means that $\forall v (z_2 \leqslant v \leqslant x \Longrightarrow \lambda u_2'(v) - u_1'(v) < 0)$, thus

$$\lambda u_2(x) - u_1(x) = \lambda u_2(y) - u_1(y) + \int_y^x (\lambda u_2'(v) - u_1'(v))\,\mathrm{d}v < 0$$

which contradicts to the definition of $\lambda = u_1(x)/u_2(x)$. Thus the earlier case $\forall z\ \big(0 < z < y \Longrightarrow \lambda u_2'(z) - u_1'(z) = 0\big)$ must be true. Consequently, together with $u_1(0) = u_2(0) = 0$, we get $\forall z\ (0 < z < y \Longrightarrow \lambda u_2(z) - u_1(z) = 0)$.

For $\forall z,\ y < z < x$, applying the inequality result in Part I, we have

$$\frac{u_1(z)}{u_1(y)} \geqslant \frac{u_2(z)}{u_2(y)} = \frac{\lambda u_2(z)}{\lambda u_2(y)} \Longrightarrow \frac{u_1(z)}{\lambda u_2(z)} \geqslant \frac{u_1(y)}{\lambda u_2(y)} = 1 \Longrightarrow \lambda u_2(z) - u_1(z) \leqslant 0.$$

Also, Part I says that $\forall z\ (0 < z < x \Longrightarrow \lambda u_2(z) - u_1(z) \geqslant 0)$. Therefore, $\forall z\ (y < z < x, \Longrightarrow \lambda u_2(z) - u_1(z) = 0)$. Thus, $\forall z\ (0 < z < x \Longrightarrow \lambda = u_1(z)/u_2(z))$. □

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